USING THE HODOGRAPH METHOD TO INVESTIGATE NONLINEAR EFFECTS IN PLASMAS

PMM Vol. 33, №2, 1969, pp. 355-358

O. M. SAPUNKOVA
(Saratov)

(Received June 3, 1968)

The relationship between the current density j in a plasma and the electric field intensity E is nonlinear in the case of strong electric fields. The nonlinearity of the initial equations engenders difficulties in computing the current density vector field in the medium. These can be eliminated by investigating the electrodynamics equations in the hodograph plane where they are linear. Emets [1] transformed the electrodynamics equations from the physical into the hodograph plane in the case of two-dimensional steady fields for an arbitrary relationship between the electrical conductivity σ and the current density. Investigation of electrodynamic processes in plasmas for a specific $\sigma(j)$ determined by the state of the plasma is a natural extension of his findings. We shall write out the electrodynamics equations in the hodograph plane for a highly ionized plasma with allowance for elastic electron-ion collections. We shall then separate variables to obtain the general solutions of these equations and use these solutions to solve the problem of the current in an infinite vessel with nonconductive walls which is analogous to the problem of gas escape from an infinite vessel [2].

Since practical considerations make it impossible to allow for the effects of all factors (ionization, radiation, inelastic collisions, etc.) on the function $\sigma(j)$ in strong electric fields, we propose that this function be approximated by a power (or exponential) function in which the constant parameters mus be determined experimentally. In the present paper we solve the problem of the current in a vessel with nonconductive walls for a power function $\sigma(j)$. An exponential function $\sigma(j)$ makes it possible to formulate mixed boundary value problems for the electrodynamics equations.

1. The following relations are valid [1 and 3] in a highly ionized plasma (with allowance for elastic electron-ion collisions):

$$\sigma = \sigma_0\left(1 + \frac{v^2}{v_0^2}\right)^\gamma, \quad v_{\max}^2 = v_0^2 + v^2, \quad \mathbf{j} = Nev$$
$$\left(v_{\max}^2 = \frac{3kTe}{M}, \quad v_0^2 = \frac{3kT}{M}\right) \tag{1.1}$$

Here T, T_e are the effective temperatures of the heavy particles and the electrons ; M is the mass of the heavy particle ; k is the Boltzmann constant ; v_0^2, $v_{\max}^2$ are proportional to the local kinetic energies of the plasma and the electrons ; σ_0 is the electrical conductivity in the plasma when $T_e = T$; N is the electron concentration ; v is the directional electron velocity, e is the electron charge ; γ is a constant.

Assuming that T = const, M = const, σ_0 = const, $\gamma > 1/2$, N = const in a given state of the plasma, we can write

$$\Gamma^2 = \frac{j}{\sigma}\frac{d\sigma}{dj} = \frac{2\gamma j^2}{j^2 + j_0^2}\ (j_0^2 = N^2e^2v_0^2 = \text{const}) \tag{1.2}$$

Let us introduce the equations for the electrical current in the hodograph plane [1],

$$\frac{\partial^2 P}{\partial j^2} + \frac{1 - \Gamma^2}{j^2}\frac{\partial^2 P}{\partial \theta^2} + \frac{1 + \Gamma^2}{j}\frac{\partial P}{\partial j} = 0 \tag{1.3}$$

$$\frac{\partial^2 Q}{\partial j^2}+\frac{1-\Gamma^2}{j^2}\frac{\partial^2 Q}{\partial\theta^2}-\frac{d}{dj}\ln\left[\frac{\sigma}{j}(1-\Gamma^2)\right]\frac{\partial Q}{\partial j}=0 \qquad \text{(cont)}$$

Here θ is the angle between the vector $\mathbf{j}$ and the x-axis; the potential stream function P and the force stream function Q are given by the following expressions:

$$j_x=j\cos\theta=\frac{\sigma}{\sigma_0}\frac{\partial P}{\partial x}=\frac{\partial Q}{\partial y},\quad j_y=j\sin\theta=\frac{\sigma}{\sigma_0}\frac{\partial P}{\partial y}=-\frac{\partial Q}{\partial x}\quad\left(\begin{matrix}x=x(j,\theta)\\ y=y(j,\theta)\end{matrix}\right) \qquad (1.4)$$

Recalling (1.2), we can rewrite expressions (1.3) as

$$2\xi(1-\xi)^2\frac{\partial^2 P}{\partial\xi^2}+\frac{1-2\gamma\xi}{2\xi}\frac{\partial^2 P}{\partial\theta^2}+2(1-\xi)(1+\gamma\xi-2\xi)\frac{\partial P}{\partial\xi}=0 \qquad (1.5)$$

$$2\xi(1-\xi)^2\frac{\partial^2 Q}{\partial\xi^2}+\frac{1-2\gamma\xi}{2\xi}\frac{\partial^2 Q}{\partial\theta^2}+\frac{2(1-\xi)}{1-2\gamma\xi}(1-\gamma\xi-2\xi+2\gamma\xi^2+2\gamma^2\xi^2)\frac{\partial Q}{\partial\xi}=0 \qquad (1.6)$$

$$\xi=\frac{\Gamma^2}{2\gamma}=\frac{j^2}{j^2+j_0^2} \qquad (1.7)$$

It is clear that $\xi<1$. Equations (1.5) and (1.6) are elliptic for $\xi<{}^1/_2\gamma^{-1}<1$.

2. Separation of variables in Eqs. (1.5) and (1.6) yields the following result. Expressing the function $P(\xi,\theta)$ as

$$P_\lambda(\xi,\theta)=f_\lambda(\xi)T_\lambda(\theta) \qquad (2.1)$$

we infer from Eq. (1.5) that $T_\lambda(\theta)$ satisfies the relation $T''+\lambda^2T=0$ (λ is an arbitrary parameter which is determined as an eigenvalue of the Sturm-Liouville problem in solving the boundary value problem). Hence,

$$T_\lambda(\theta)=C_1e^{i\lambda\theta}+C_2e^{-i\lambda\theta}\ \text{for}\ \lambda\neq0,\ T_0(\theta)=A+B\theta\ \text{for}\ \lambda=0,\ (A,B,C_1,C_2=\text{const}) \qquad (2.2)$$

For $\lambda\neq0$ the function $f_\lambda(\xi)$ satisfies Riemann's P-equation [4]

$$f=P\left\{\begin{matrix}0 & \infty & 1 & \\ {}^1/_2\lambda & 0 & {}^1/_2(\gamma+\gamma_\lambda) & \xi\\ -{}^1/_2\lambda & 1-\gamma & {}^1/_2(\gamma-\gamma_\lambda) & \end{matrix}\right\}$$

and can be expressed in the form

$$\begin{aligned}f_\lambda(\xi)=\xi^{1/2\lambda}(\xi-1)^{1/2(\gamma+\gamma_\lambda)}\{C_3F({}^1/_2(\lambda+\gamma)+{}^1/_2\gamma_\lambda,\ 1+{}^1/_2(\lambda-\gamma)+\\ +{}^1/_2\gamma_\lambda,\ 1+\lambda,\ \xi)+C_4\xi^{-\lambda}F({}^1/_2(\gamma-\lambda)+{}^1/_2\gamma_\lambda,\ 1-{}^1/_2(\lambda+\gamma)+\\ +{}^1/_2\gamma_\lambda,\ 1-\lambda,\ \xi)\};\quad (\gamma_\lambda=\sqrt{\gamma^2-(2\gamma-1)\lambda^2})\end{aligned} \qquad (2.3)$$

Here C_3, C_4 are constants, λ is not an integer, and $F(a,b,c,\xi)$ is the hypergeometric function

$$f_0(\xi)=C_5+C_6\int\frac{(1-\xi)^{\gamma-1}}{\xi}d\xi\qquad\text{for}\ \lambda=0 \qquad (2.4)$$

Similarly, we can express the function $Q(\xi,\theta)$ in the form

$$Q_\lambda(\xi,\theta)=\varphi_\lambda(\xi)\psi_\lambda(\theta) \qquad (2.5)$$

$$\psi_\lambda(\theta)=\frac{dT_\lambda(\theta)}{d\theta},\quad \varphi_\lambda(\xi)=\frac{2\xi}{\lambda^2(1-\xi)^{\gamma-1}}\frac{df_\lambda(\xi)}{d\xi}\qquad\text{for}\ \lambda\neq0 \qquad (2.6)$$

$$\psi_0(\theta)=C_7+C_8\theta,\quad \varphi_0(\xi)=C_9+C_{10}\int\frac{1-2\gamma\xi}{\xi(1-\xi)^{\gamma+1}}d\xi\qquad\text{for}\ \lambda=0 \qquad (2.7)$$

$$(C_7,\ C_8,\ C_9,\ C_{10}=\text{const})$$

3. Separation of variables can be used to solve several boundary value problems, one of which is analogous to the problem of gas escape from an infinite vessel [2].

Let us consider an infinite vessel with symmetric nonconductive walls filled with a highly ionized plasma. An opening in the vessel is fitted with a cathode of width $BB'=2b$ (Fig. 1). We are to determine the current density field in the vessel assuming that the process is steady and two-dimensional, that the charge density is the same at all points of the electrode, and that $\xi_1 < {}^1/_2\gamma^{-1}$ (ξ_1 corresponds to the current density j_1 at the vessel outlet); the layer adjacent to the electrode is not taken into account.

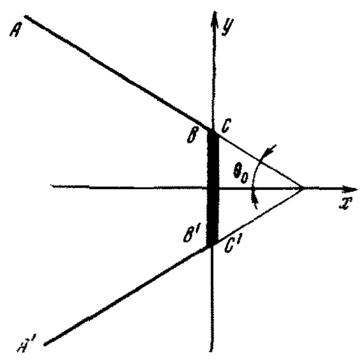

Fig. 1

Carried into the hodograph plane ($\xi_x = \xi\cos\theta$, $\xi_y = \xi\sin\theta$), the problem can be stated as follows: we are to determine the function $Q(\xi, \theta)$ in the sector $0 < \xi < \xi_1, -\theta_0 < \theta < \theta_0$ (Fig. 2) from the boundary conditions

$$Q(\xi, \theta_0) = -{}^1/_2 I \quad (0 < \xi \leqslant \xi_1)$$
$$Q(\xi, -\theta_0) = {}^1/_2 I \quad (0 < \xi \leqslant \xi_1) \tag{3.1}$$
$$Q(\xi_1, \theta) = -{}^1/_2 I \quad (0 < \theta \leqslant \theta_0)$$
$$Q(\xi_1, \theta) = {}^1/_2 I \quad (-\theta_0 \leqslant \theta < 0)$$

where I is the current strength at the vessel outlet,

$$I = 2j_1 b = \frac{2b\xi_1^{1/2} j_0}{(1-\xi_1)^{1/2}} \tag{3.2}$$

Moreover, by symmetry,

$$Q(\xi, 0) = 0 \quad (0 < \xi < \xi_1) \tag{3.3}$$

Separation of variables yields the solution of the above boundary value problem in a form similar to the solution of Chaplygin's jet problem, i.e.

$$Q(\xi, \theta) = -\frac{I\theta}{2\theta_0} - \frac{I}{\pi}\sum_{\lambda=1}^{\infty}\frac{1}{\lambda}\frac{z_\lambda(\xi)}{z_\lambda(\xi_1)}\sin\frac{\lambda\pi\theta}{\theta_0} \tag{3.4}$$

where

$$z_\lambda(\xi) = \xi^{\lambda+1}\,\mathrm{Re}\left\{(1-\xi)^{1-{}^1/_2\gamma+{}^1/_2\gamma_\lambda}\left[\left(\frac{\lambda}{\xi} + \frac{1}{2}\frac{\gamma + \sqrt{\gamma^2-(2\gamma-1)4\lambda^2}}{1-\xi}\right)\times\right.\right.$$
$$\times F(\lambda + {}^1/_2\gamma + {}^1/_2\gamma_\lambda, \lambda + 1 - {}^1/_2\gamma + {}^1/_2\gamma_\lambda;\ 1+2\lambda;\ \xi) +$$
$$+ (1+2\lambda)^{-1}(\lambda + {}^1/_2\gamma + {}^1/_2\gamma_\lambda)(\lambda + 1 - {}^1/_2\gamma + {}^1/_2\gamma_\lambda)\,F(1 + \lambda + {}^1/_2\gamma +$$
$$\left.\left. + {}^1/_2\gamma_\lambda,\ 2 + \lambda - {}^1/_2\gamma + {}^1/_2\gamma_\lambda;\ 2+2\lambda;\ \xi)\right]\right\} \tag{3.5}$$

$$\gamma_\lambda = \sqrt{\gamma^2 - (2\gamma-1)4\lambda^2}$$

Relation (3.4) enables us to determine the current density j at any point inside the vessel.

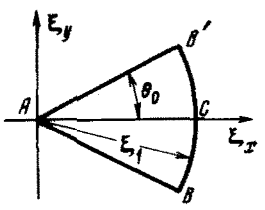

Fig. 2

4. Allowance for all the factors affecting the relationship between σ and j (electron-electron collisions, ionization, radiation, etc.) is very difficult and practically impossible for plasmas with any degree of ionization, and even for highly ionized plasmas. It is therefore expedient to approximate the function $\sigma(j)$ by some curve, e.g. in the form $(j) = aj^\alpha$, where the parameters a and α must be determined experimentally and are assumed constant within given intervals of variation of j. With the function specified in this way the electrodynamics equations in the hodograph plane [Eqs. (1.3)] become

$$\frac{\partial^2 P}{\partial j^2} + \frac{1-\alpha}{j^2}\frac{\partial^2 P}{\partial\theta^2} + \frac{1+\alpha}{j}\frac{\partial P}{\partial j} = 0 \tag{4.1}$$

$$\frac{\partial^2 Q}{\partial j^2} + \frac{1-\alpha}{j^2}\frac{\partial^2 Q}{\partial\theta^2} + \frac{1-\alpha}{j}\frac{\partial Q}{\partial j} = 0 \tag{4.2}$$

Separation of variables yields the solution of Eq. (4.2) in the form where

$$Q_\lambda(j,\theta) = \varphi_\lambda(j)\,\psi_\lambda(\theta)$$

$$\varphi_\lambda(j) = j^{1/2\alpha}\,(C_1 j^{+\alpha_\lambda} + C_2 j^{-\alpha_\lambda})$$

$$\psi_\lambda(\theta) = C_3 e^{i\lambda\theta} + C_4 e^{-i\lambda\theta}\quad (\alpha_\lambda = \sqrt{1/4\alpha^2 + \lambda^2(1-\alpha)}) \tag{4.3}$$

for $\lambda \neq 0$, and

$$\varphi_0(j) = A + Bj^\alpha,\ \psi_0(\theta) = C + D\theta \quad (C_1, C_2, C_3, C_4, A, B, C, D = \text{const})$$

for $\lambda = 0$.

The parameter λ is determined as an eigenvalue of the Sturm-Liouville problem. The solution of the problem of Sect. 3 in this case becomes

$$Q(j,\theta) = -\frac{I}{2\theta_0}\theta - \frac{I}{\pi}\sum_{\lambda=1}^{\infty}\frac{1}{\lambda}\left(\frac{j}{j_1}\right)^{1/2\alpha+\alpha_\lambda}\sin\frac{\pi\lambda\theta}{\theta_0} \tag{4.4}$$

where j_1 is the current density at the vessel outlet and $\alpha < 1$. If the function $\sigma(j)$ can be expressed in the form $\sigma = be^{i\beta}$, where b and β must be determined experimentally and are constant within given intervals of variation of j, then Eqs. (1.3) become

$$\frac{\partial^2 P}{\partial j^2} + \frac{1 - j\beta}{j^2}\frac{\partial^2 P}{\partial\theta^2} + \frac{1 + j\beta}{j}\frac{\partial P}{\partial j} = 0$$

$$\frac{\partial^2 Q}{\partial j^2} + \frac{1 - j\beta}{j^2}\frac{\partial^2 Q}{\partial\theta^2} + \frac{1 - j\beta + j^2\beta^2}{j(1 - j\beta)}\frac{\partial Q}{\partial j} = 0 \tag{4.5}$$

Unlike relations (4.1) and (4.2), Eqs. (4.5) do not exclude smooth transition of the process from the ellipticity to the hyperbolicity domain for a constant α. This makes it possible to formulate mixed boundary value problems for the electrodynamics equations in the hodograph plane. We note, however, that A. G. Kulikovskii and S. A. Regrer (PMM Vol. 32, №4, 1968) demonstrated the impossibility of formulating boundary value problems in purely hyperbolic domains.

The author is grateful to S. V. Fal'kovich for his useful remarks.

BIBLIOGRAPHY

1. Emets, Iu. P., The hodograph method in electrodynamics of continuous nonlinearly conducting media. PMM Vol. 31, №6, 1967.

2. Chaplygin, S. A., On gas jets. In: Works, Vol. 2, Gostekhizdat, Moscow, 1948.

3. Ginsburg, V. L., Propagation of Electromagnetic Waves in Plasmas, 2nd edition. Moscow, "Nauka", 1967.

4. Whittaker, E. T. and Watson, G. N., A Course in Modern Analysis, (Russian translation) 2nd edition. Fizmatgiz, Moscow, 1963.

Translated by A. Y.